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The summability of the $1/N$ expansion in a simplified O(N) model

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Abstract. We discuss the $1/N$ expansion of an O(N) quartic single-mode model. The imposition of boundary conditions by the $1/N$ expansion is explicitly displayed, using the exact solutions.

In particular, it is shown that in the negative mass^2 case the $1/N$ expansion *is* Borel summable whereas the λ(and $\hbar$) expansions are not. The applicability of these results to more realistic models (i.e. as the first term of a strong coupling approximation scheme) is discussed.

1. Introduction

In the past few years there has been considerable interest in approximation schemes for computing Green functions of local field theories that are different to the Feynman diagram perturbation series in the coupling constant λ of the theory. The motivation for these schemes has been the realisation that the λ-perturbation series can correspond to expansion about the wrong vacuum. In particular, a possibility discussed by many authors has been the *dynamical* breaking of the symmetry of a local field theory by purely quantum-mechanical effects, a circumstance which (in general) is not amenable to finite-order perturbation theory calculations in λ.

For the majority of cases discussed in the literature, the approximations adopted for examining the possibility of solutions different from λ-perturbation solutions are based on the Schwinger–Dyson branching equations for the connected Green functions (appropriately approximated). One of the solutions to these equations will be expected to be close to the λ-perturbation series. The situation becomes interesting if, in addition to this solution, other solutions exist of an entirely different type, which we will call the non-λ-perturbative solutions.

Merely to have a (necessarily *non-linear*) approximation‡ (Nambu and Jona-Lasinio 1961, Jackiw and Johnson 1973, Cornwall and Norton 1973, Englert *et al* 1974, 1975, Caianiello and Marinaro 1971, Caianiello *et al* 1971) that gives non-λ-perturbative Green functions is insufficient. It is necessary to imbed this approximation

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‡ A typical approximation would be a Hartree–Fock type approximation. Nambu and Jona-Lasinio (1961), Jackiw and Johnson (1973), Cornwall and Norton (1973), Englert *et al* (1974, 1975) and Caianiello and Marinaro (1971) give some typical solutions to this approximation displaying non-λ-perturbative solutions. We note that the existence of more than one solution is not necessarily connected to dynamical symmetry breaking, although if there is no change in symmetry the physical conclusions will be less remarkable.

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in an *approximation scheme* which, if soluble, it is hoped will give more and more accurate representations of (all solutions for) the connected Green functions.

The simplest such approximation schemes correspond to expansion of the Green function branching equations in some parameter, ϵ say, that is different from λ. This parameter may be just an artifact of the approximation scheme, a book-keeping device to be set unity at the end of the calculation (Wong and Guralnik 1971, Snyderman 1977). Alternatively it might be another parameter of the theory. A typical example is the $\epsilon = N^{-1}$ expansion of some O(N) (Schnitzer 1974a,b, Coleman *et al* 1975, Kobayashi and Kugo 1975, Abbott *et al* 1976, Rivers 1976) or U(N) invariant theory.

In practice, for realistic theories, successive approximations get more and more difficult and (just as in λ-perturbation expansions) only the first few steps are technically practicable (Caianiello *et al* 1971, Root 1974). Thus the important problem of the possibility of reconstructing the Green functions *uniquely* from the ϵ expansions is often ignored. Of course, if the ϵ expansions do not permit reconstruction of unique Green functions the approximation scheme is worthless.

It is to this problem that we address ourselves. One approach that could be adopted would be first to evaluate the large-order terms of any ϵ-expanded Green function and determine whether all expansions (assumed asymptotic series in ϵ) were Borel summable. This is the standard approach to the λ-perturbation series for Green functions (Brézin *et al* 1977, Itzykson *et al* 1977a,b, Parisi 1977), whose unique summability is also of importance. Borel summability is a useful guide, since the failure of Borel summability certainly implies non-unique reconstruction of Green functions, although the converse is not true. Neglecting problems with infinite renormalisation much work has been done on the λ-perturbation series for Euclidean field theories and it might be thought that the tactics applied there could be usefully transferred to the problem in hand. Certainly we shall be unable to accommodate infinite renormalisation and we shall only consider Euclidean field theories that have been fully regularised.

As a first step, however, we see that a very different approach will give us information on the summability of ϵ expansions for Green functions for the limited class of approximation schemes in which the explicit theoretical input is *only* the Schwinger–Dyson equations (as it would be for purely scalar theories). We restrict ourselves to this class of schemes.

Consider the following facts.

(i) The ϵ expansions are determined (for these schemes) by iterative approximation to the *non-linear* Schwinger–Dyson branching equations (well defined for a regularised theory) for the *connected* Green functions. The exact equations are, of course, equivalent to the *linear* Schwinger–Dyson equations for the *non-connected* Green functions. In the absence of further constraints (and we have restricted ourselves to situations where there are none) the solutions to these latter equations are *not* unique, depending continuously upon arbitrary functionals, specifying the boundary conditions.

(ii) On the other hand, in solving for the ϵ expansion by iteratively modifying the *non-linear* equations, at each step we have a discrete number (up to global symmetry invariance) of exact solutions.

Superficially we seem to have a paradox, but it is not difficult to guess how the problem of matching continuous infinities of solutions to a discrete number of solutions is resolved. Firstly, it is not surprising if solving linear equations in a non-linear way corresponds to imposing weak boundary conditions upon them, although it is difficult to state *a priori* what these conditions will be. Of course, if these implicit boundary conditions were complete there would be no problem. However, in the likely event of them being incomplete the residual ambiguity can be accommodated by one or more of the ϵ expansions not being uniquely summable to a particular solution of the Schwinger–Dyson equations.

In the next few sections of this paper we shall show how these ideas are effected in a concrete (albeit non-realistic) example of an O(N) invariant $\lambda\phi^4$ theory in a single-mode approximation (or equivalently, in zero dimensions), taking $\epsilon = N^{-1}$ as the expansion parameter. This model, which essentially counts all Feynman diagrams with equal weight, possesses a non-λ-perturbative solution to the Schwinger–Dyson branching equations for the connected Green functions. Because it is so simple as to be exactly soluble it is possible to examine the problems of summability and to display explicitly the boundary conditions which the approximation scheme imposes.

So far we have only considered the Schwinger–Dyson equations in determining the uniqueness of summability of the ϵ expansion. This is not adequate, since many solutions to these equations will be 'unphysical'. Imposing 'physicality' constraints (e.g. no 'tachyons') will restrict us to a subset of the general solutions, and it is ultimately in this subset that uniqueness has to be considered. Again, in this model it is easy to demonstrate this.

The plan of the paper is as follows. Since we expect the $\epsilon = N^{-1}$ expansions due to the iterative truncation of the Schwinger–Dyson equation to be less general than the N^{-1} expansion of the exact solution we display them in the next section. In § 3 we construct the exact solution and compare their N^{-1} expansions to those already obtained. The boundary conditions which the approximation scheme imposes are found to be implied by the renormalisation group equations. In § 5 we consider 'physical' constraints upon the exact solutions, and in §§ 4 and 6 we examine the λ and N^{-1} summability and relate them.

Finally, we briefly examine the extent to which this simple model is relevant to more realistic situations, in particular, to 'strong coupling' expansions for O(N) lattice theories.

2. The O(N) model in the single-mode approximation and its N^{-1} expansion

The theory that our toy model is motivated by is that described by the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a - \tfrac{1}{2}m_0^2\varphi_a^2 - \frac{\lambda_0}{4N}(\varphi_a^2)^2 \tag{2.1}$$

where the φ_a ($a = 1, 2, \ldots, N$) provide an N-dimensional scalar field representation of O(N). When m_0^2 is negative the O(N) symmetry is, of course, spontaneously broken classially. Until stated otherwise, we shall assume that m_0^2 is positive.

Until the last section we shall be examining the expansion of the Green functions of the single-mode approximation to the Euclidean theory of equation (1.1). Combinatorially we can interpret the single-mode approximation as (a) the zero-dimensional theory based on equation (2.1), (b) the non-local approximation in which the ϕ_a fields propagate in a box independently of space-time, or (c) the static ultra-local approximation on a grid (Caianiello *et al* 1965, Caianiello and Scarpetta 1974). This latter interpretation permits our discussion to be relevant to strong-coupling expansions for O(N) invariant lattice theories. We defer this to the last section.

In order to generate the N^{-1} expansions for the connected Green functions (the quantities of 'physical' interest) we need the non-linear equations relating them. These are derived from the linear equations for the non-connected Green functions, which in turn are derived from the linear Schwinger–Dyson equation for the generating function $Z_0[q_a]$, where the q_a (of which the space-time structure is irrelevant in the single-mode approximation) are the sources coupled to the 'fields' ϕ_a.

This equation is the third-order linear equation

$$m_0^2\frac{\partial Z_0}{\partial q_a}=q_aZ_0-\frac{\lambda_0}{N}\frac{\partial^3 Z_0}{\partial q_a\partial q_b^2}. \tag{2.2}$$

As a consequence of the O(N) symmetry of the model Z is only a function of $Q^2=q_a^2$. Equation (2.2) now takes the form

$$m_0^2\frac{\partial Z_0}{\partial Q}=QZ_0-\frac{\lambda_0}{N}\left[\frac{\partial^3 Z_0}{\partial Q^3}+\frac{N-1}{Q}\left(\frac{\partial^2 Z_0}{\partial Q^2}-\frac{1}{Q}\frac{\partial Z_0}{\partial Q}\right)\right]. \tag{2.3}$$

Expanding Z in terms of the non-connected Green functions τ_n of the model as

$$Z_0=\sum_n \tau_n Q^n/n! \tag{2.4}$$

with normalisation $\tau_0=1$ we see that, for $N\neq 1$, $\tau_1=0$, whence there is no solution which breaks the O(N) symmetry.

Before examining the general solution of equation (2.3) we shall develop the $1/N$ expansion of the connected Green functions, to show how the $1/N$ expansion gives a *discrete number* of approximate solutions to a *linear* differential equation with as yet arbitrary boundary conditions.

If

$$W_0(Q)=\ln Z_0(Q) \tag{2.5}$$

is the generating function for the connected Green functions Γ_n, equation (2.3) becomes, in terms of W, the highly non-linear equation

$$m_0^2\frac{\partial W_0}{\partial Q}=Q-\frac{\lambda_0}{N}\left\{\frac{\partial^3 W_0}{\partial Q^3}+3\frac{\partial^2 W_0}{\partial Q^2}\frac{\partial W_0}{\partial Q}+\left(\frac{\partial W_0}{\partial Q}\right)^3+\frac{N-1}{Q}\left[\frac{\partial^2 W_0}{\partial Q^2}+\left(\frac{\partial W_0}{\partial Q}\right)^2-\frac{1}{Q}\frac{\partial W_0}{\partial Q}\right]\right\}. \tag{2.6}$$

The Γ_n are then given by

$$W_0=\sum_n \Gamma_n Q^n/n!. \tag{2.7}$$

For $N\neq 1$ all Γ_{2p+1} (p integer) are zero, and the Γ_{2p} satisfy the expected non-linear relations, of which the first

$$\Gamma_2=\frac{1}{m_0^2}-\frac{\lambda_0}{m_0^2}\left(\frac{N+2}{N}\Gamma_2^2+\frac{N+2}{3N}\Gamma_4\right) \tag{2.8}$$

is exemplary. To avoid confusion in generating N^{-1} expansions for $\Gamma_2, \Gamma_4, \ldots$, etc, we will make the implicit N dependence of equation (2.6) explicit by setting

$$W_0=N\omega$$

and

$$Q=m_0\sqrt{N}q. \tag{2.9}$$

Introducing the dimensionless parameter $\lambda = \lambda_0 m_0^{-4}$ equation (2.6) now becomes

$$\frac{\partial\omega}{\partial q} = q - \lambda\left[\left(\frac{\partial\omega}{\partial q}\right)^3 + \frac{N-1}{Nq}\left(\frac{\partial\omega}{\partial q}\right)^2 + \frac{3}{N}\frac{\partial^2\omega}{\partial q^2}\frac{\partial\omega}{\partial q} + \frac{N-1}{N^2 q}\left(\frac{\partial^2\omega}{\partial q^2} - \frac{1}{q}\frac{\partial\omega}{\partial q}\right) + \frac{1}{N^2}\frac{\partial^3\omega}{\partial q^3}\right]. \tag{2.10}$$

Taking q to be O(1) it follows that $\omega(q)$ and its derivatives are all O(1)†. An expansion for ω in terms of N as

$$\omega(q)_N = \sum_p \omega^{(p)}(q)/N^p \tag{2.11}$$

can then be made, and solved iteratively order by order.

It is straightforward to show that the non-linear equation (1.10) for $\omega(q)$ gives just *two* solutions for $\omega(q)_N$.

First, in leading order

$$\frac{\partial\omega^{(0)}}{\partial q} = q - \lambda\left[\left(\frac{\partial\omega^{(0)}}{\partial q}\right)^3 + \frac{1}{q}\left(\frac{\partial\omega^{(0)}}{\partial q}\right)^2\right]. \tag{2.12}$$

This equation gives rise to *two* solutions for $\omega^{(0)}$ vanishing at $q = 0$, which we denote $\omega^{(0)\pm}$.

The next to leading term $\omega^{(1)}$ is fixed uniquely as a function of $\omega^{(0)}$ by the equation

$$\frac{\partial\omega^{(1)}}{\partial q} = -\lambda\left(\frac{\partial^2\omega^{(0)}}{\partial q^2}\frac{\partial\omega^{(0)}}{\partial q} + \frac{1}{q}\frac{\partial^2\omega^{(0)}}{\partial q^2} - \frac{1}{q^2}\frac{\partial\omega^{(0)}}{\partial q}\right)\left[1 + 3\lambda\left(\frac{\partial\omega^{(0)}}{\partial q}\right)^2\right]^{-1} \tag{2.13}$$

with solutions $\omega^{(1)\pm}$. The pattern continues, with two possible solutions for higher corrections $\omega^{(p)\pm}$, giving two asymptotic series

$$\omega_N^{\pm} = \sum_{p=0}^{\infty} \omega^{(p)\pm}/N^p. \tag{2.14}$$

If we wish, from $\omega_N^{\pm}$ we can now construct two functions W, $W_N^{\pm}(Q^2)$ and hence $Z_N^{\pm}(Q^2)$. From $Z_N^{\pm}$ we can, in turn, construct the two N^{-1} expansions $(\tau_{2p}^{\pm})_N$ for the non-connected Green functions. It is these Green functions that are simplest to compare to the exact solutions of the next section.

We conclude this section with several disparate comments on the N^{-1} expansion. Firstly, we note that, to leading order, the equation for Γ_2 is

$$\Gamma_2^{(0)} = \Delta^{(0)} - \lambda_0\Delta^{(0)}\Gamma_2^{(0)}\Gamma_2^{(0)} \tag{2.15}$$

where $\Delta^{(0)} = m_0^{-2}$ is the free two-point function of the model. Equation (2.15) has the diagrammatic interpretation of figure 1, the conventional Hartree–Fock-like approximation. The solutions to equation (2.15) are

$$2\Gamma_2^{(0)\pm} = [-m_0^2 \pm (m_0^4 + 4\lambda_0)^{1/2}]/\lambda_0. \tag{2.16}$$

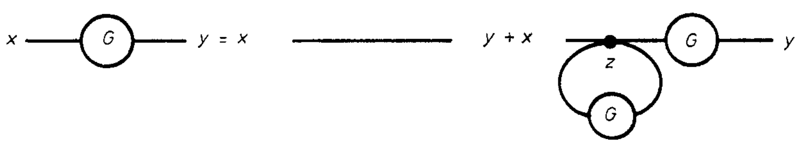


Figure 1. A diagrammatic representation of the truncated form of the Schwinger–Dyson equation.

† Whence $\Gamma_p = O(N^{1-p/2})$.

We note that these two solutions are also easily seen in direct calculation of the effective potential V using an auxiliary field χ coupling to φ_a^2. This gives (to leading order in N^{-1})†

$$V(\varphi_a^2, \chi) = -\frac{N\chi^2}{4\lambda_0} + \tfrac{1}{2}\chi\varphi_a^2 + \frac{Nm_0^2\chi}{2\lambda_0} + \frac{N}{2}\ln\left(\frac{\chi}{m_0^2}\right). \tag{2.17}$$

The potential $V(\varphi_a^2)$ is determined by evaluating $V(\varphi_a^2, \chi)$ for

$$\frac{\partial V}{\partial \chi} = 0 = -\frac{N\chi}{2\lambda_0} + \tfrac{1}{2}\varphi_a^2 + \frac{Nm_0^2}{2\lambda_0} + \frac{N}{2\chi} \tag{2.18}$$

which has two branches, with solutions $\chi_\pm(\varphi_a^2)$. The extrema of $V(\varphi_a^2)$ also satisfy

$$\partial V/\partial\varphi_a = \varphi_a\chi = 0. \tag{2.19}$$

Since $\chi = 0$ is not compatible with equation (2.18) we must choose $\varphi_a = 0$ (as observed earlier) whence equation (2.18) becomes

$$\chi = m_0^2 + \lambda_0\chi^{-1}. \tag{2.20}$$

Since χ plays the role of the $(\text{mass})^2$ of the fields at the extrema, it can be identified with $(\Gamma_2^{(0)})^{-1}$. Equation (2.20) then becomes identical to equation (2.15). We note that $\chi_\pm$ are of opposite sign. With the above interpretation negative χ is 'unphysical' and should be excluded. We shall have more to say about this later.

Finally, we stress that it is the summability of the N^{-1} expansions of the solutions indicated above that concerns us. We expect the exact solutions for Γ_{2p} to permit further N^{-1} expansions different from the above as the boundary conditions change. We are not interested in these.

3. Comparison to the exact solution

Let us now construct the general solution (regular at $q_a = 0$) to equation (2.2), to establish its relation to the N^{-1} expansion of the previous section. This is (observing the invariance of the equation under $m_0^2 \to -m_0^2$, $q_a \to \pm iq_a$)

$$Z_0(q_a, m_0^2, \lambda_0, N) = \alpha(m_0^2, \lambda_0, N)\bar{Z}_+(q_a, m_0^2, \lambda_0, N) + \beta(m_0^2, \lambda_0, N)\bar{Z}_-(q_a, m_0^2, \lambda_0, N) \tag{3.1}$$

where

$$\bar{Z}_-(\pm iq_a, -m_0^2, \lambda_0, N) = \bar{Z}_+(q_a, m_0^2, \lambda_0, N)$$
$$= \int_{-\infty}^{\infty} \prod_1^N dx_a \exp\left(-\frac{\lambda_0}{4N}(x_a^2)^2 - \frac{m_0^2}{2}x_a^2 + x_a q_a\right) \tag{3.2}$$

and α, β are arbitrary functions of m_0^2, λ_0, N. Normalising Z to unity at $q_a = 0$ gives Z in terms of *one arbitrary* function

$$\eta(m_0^2, \lambda_0, N) = \beta/\alpha \tag{3.3}$$

† Obtained directly by saddle-point methods, or diagrammatically via the prescription of Schnitzer (1974a,b), Coleman *et al* (1975), Kobayashi and Kugo (1975), Abbott *et al* (1976) and Rivers (1976). In D dimensions $\ln(\chi/m_0^2)$ is replaced by $\int[d^Dk/(2\pi)^D]\ln(k^2+\chi)$.

or equivalently, τ_2. Thus, for example, τ_4 is given in terms of the *arbitrary* τ_2 by

$$\frac{\lambda_0}{3N}(N+2)\tau_4+m_0^2\tau_2-1=0 \tag{3.4}$$

and so on.

This is to be contrasted with the N^{-1} expansion in which τ_2 is *determined* (order by order in N^{-1}) from the non-linear equations (2.15), (2.13), etc.

Integrating over the angular variables we get

$$\bar{Z}_{\pm}(Q, m_0^2, N)=\pi^{N/2}\int_{C_\pm} \mathrm{d}x^2\frac{(x^2)^{(N/2)-1}}{(\frac{1}{2}|x||Q|)^{(N/2)-1}}I_{(N/2)-1}(|x||Q|)\exp\left[-\left(\frac{\lambda_0}{4N}x^4+\tfrac{1}{2}m_0^2x^2\right)\right], \tag{3.5}$$

where C_+ is $[0, \infty)$ and C_- is $(-\infty, 0]$. This gives

$$\tau_{2p}=\frac{\alpha(m_0^2, \lambda_0, N)\bar{\tau}_{2p}^{+}+\beta(m_0^2, \lambda_0, N)\bar{\tau}_{2p}^{-}}{\alpha(m_0^2, \lambda_0, N)\bar{\tau}_0^{+}+\beta(m_0^2, \lambda_0, N)\bar{\tau}_0^{-}} \tag{3.6}$$

where

$$\bar{\tau}_{2p}^{\pm}=\left(\frac{N}{8\lambda_0}\right)^{p/2}\frac{(2p)!}{p!}U\left(\frac{N+2p-1}{2}, \pm(N/2\lambda)^{1/2}\right)(-1)^{p/2\mp p/2}. \tag{3.7}$$

The $U(a, x)$ are the usual parabolic cylinder functions.

What interests us in this section is how the series $(\tau_{2p}^{\pm})_N$ in N^{-1} (defined after equation (2.14)), which were obtained by an iterative approximation to the Schwinger–Dyson equations alone, follow from the general solution (3.6).

On examining the large N behaviour of $\bar{\tau}_{2p}^{\pm}$ of equation (3.7) we observe that, as $N\to\infty$ (fixed λ_0) (Caianiello *et al* 1965, Caianiello and Scarpetta 1974),

$$\frac{\bar{\tau}_{2p}^{+}}{\bar{\tau}_{2p}^{-}}=A(\lambda_0, p)\,\mathrm{e}^{-2N\theta(\lambda)}\{1+\mathrm{O}(N^{-1})\} \tag{3.8}$$

where

$$\theta(\lambda)=\tfrac{1}{4}\int_0^{1/\lambda}\mathrm{d}x\left(\frac{1}{x}+\frac{1}{4}\right). \tag{3.9}$$

That is, all the derivatives of $\tau_{2p}^{+}/\tau_{2p}^{-}$ with respect to N^{-1} are identically zero, and the ratio has (the non-unique) zero asymptotic series in N^{-1}. (Similarly $\bar{Z}_+/\bar{Z}_-$ has zero asymptotic series in N^{-1}.) For this reason the analyticity of $\bar{Z}_+$ and $\bar{Z}_-$ individually in N^{-1} is not of prime importance.

As we have said $\eta(m_0^2, \lambda_0, N)=\beta/\alpha$ is an *arbitrary* boundary condition as far as the Schwinger–Dyson equations are concerned (and the N^{-1} expansion of the previous section was generated from them *alone*). Equivalently, we can consider the arbitrary τ_2 as defining η.

Let us see how the asymptotic series in N^{-1} of τ_{2p} depends on the N behaviour of η. There are three possibilities, according to the large N behaviour of

$$K(m_0^2, \lambda_0, N)=\eta(m_0^2, \lambda_0, N)^{-1}\,\mathrm{e}^{-2N\theta(\lambda)}. \tag{3.10}$$

(i) If K has the form $K=\eta_0^{-1}k(m_0^2, \lambda_0, N)$ where k is not identically zero but has zero asymptotic series in N^{-1} and η_0 is a constant (e.g. η independent of N), τ_{2p} will have one or other of two asymptotic series in N^{-1}, as η_0 varies.

(*a*) If $\eta_0 = 0$

$$(\tau_{2p})_N \equiv (T^+_{2p})_N = \left(\frac{\bar\tau^+_{2p}}{\bar\tau^+_0}\right)_N = \left(\frac{N}{8\lambda_0}\right)^{p/2} \frac{(2p)!}{p!} \left(\frac{U[\frac{1}{2}(N+2p-1), (N/2\lambda)^{1/2}]}{U[\frac{1}{2}(N-1), (N/2\lambda)^{1/2}]}\right)_N \tag{3.11}$$

where the suffix N on a bracket denotes the N^{-1} expansion of its contents.

(*b*) If $\eta_0 \neq 0$

$$(\tau_{2p})_N \equiv (T^-_{2p})_N = \left(\frac{\bar\tau^-_{2p}}{\bar\tau^-_0}\right)_N = (-1)^p \left(\frac{N}{8\lambda_0}\right)^{p/2} \frac{(2p)!}{p!} \left(\frac{U[\frac{1}{2}(N+2p-1), -(N/2\lambda)^{1/2}]}{U[\frac{1}{2}(N-1), -(N/2\lambda)^{1/2}]}\right)_N. \tag{3.12}$$

Here we have an example of non-unique summability in that for all $\eta \neq 0$ we have the same asymptotic series.

(ii) If K has a non-zero asymptotic series in N^{-1} with finite coefficients (e.g. $\eta = a\,\mathrm{e}^{-2N\theta(\lambda)}$) equation (3.6) will give a *continuous* infinity of asymptotic series in N^{-1}.

(iii) If K^{-1} has the form $K^{-1} = \eta_0 k^{-1}(m_0^2, \lambda_0, N)$ where k^{-1} is not identically zero but has zero asymptotic series in N^{-1} and η_0 is a constant, τ_{2p} will again have one or another of two asymptotic series in N^{-1}:

$$\begin{aligned} &(a) \qquad \text{if } \eta^{-1} = 0,\ (\tau_{2p})_N = (T^-_{2p})_N \\ &(b) \qquad \text{if } \eta^{-1} \neq 0,\ (\tau_{2p})_N = (T^+_{2p})_N. \end{aligned} \tag{3.13}$$

As in (i), the non-unque summability of K guarantees the non-unique summability of $(T^+_{2p})_N$. We note that, but for $\eta = 0$ and $\eta^{-1} = 0$ these possibilities are exclusive.

Since the asymptotic series $(\tau^\pm_{2p})_N$ in N^{-1} must be two of the above series (i.e. either $(T^\pm_{2p})_N$ or elements of the continuous infinity due to the boundary condition (ii)) it is sufficient to compare leading terms in N^{-1} to determine which they are. The leading behaviour

$$(T^\pm_{2p})_N = \frac{(-1)^{p/2\mp p/2}(2p)!}{m_0^{2p} 2^p p! \lambda^{p/2}} \left[\frac{1}{2\sqrt{\lambda}} + \left(1 + \frac{1}{4\lambda}\right)^{1/2}\right]^{\mp p} \tag{3.14}$$

immediately reproduces the solutions of equation (2.16) for $\Gamma_2^{(0)\pm} = \tau_2$ whence we can identify $(T^\pm_{2p})_N$ with $(\tau^\pm_{2p})_N$.

We have thus established a correspondence between the N^{-1} expansions obtained by iteratively truncating the non-linear Schwinger–Dyson equations and the asymptotic series in N^{-1} for the exact solutions with specified boundary conditions. We see that the former N^{-1} expansion excludes possibility (ii) and we can now take this restriction as an implicit boundary condition. With this imposition τ_2 is no longer arbitrary and its asymptotic series will match the self-consistent approximations, although both solutions cannot be uniquely summable. The question arises as to whether this implicit boundary condition can be derived as a consequence of additional constraints on the solutions that are necessary for other reasons†. We shall now consider this possibility.

From dimensional arguments we see that if $\bar Z(Q)$ is the generating functional for the differently normalised unconnected Green functions

$$\kappa_p = m_0^{-N} f(\lambda) \tau_p \tag{3.15}$$

† If this were not the case then not only would the N^{-1} expansion be non-uniquely summable but there would also exist physically relevant solutions of which it was not even the large N limit. The $m_0 \to 0$ limit of this model provides an example of this.

(arbitrary f), by virtue of also satisfying the *linear* Schwinger–Dyson equation (2.2), $\bar{Z}$ satisfies

$$2m_0^2\frac{\partial \bar{Z}}{\partial m_0^2}+4\lambda_0\frac{\partial \bar{Z}}{\partial \lambda_0}=-m_0^2\frac{\partial^2 \bar{Z}}{\partial q_a^2}-\frac{\lambda_0}{N}\frac{\partial^4 \bar{Z}}{\partial q_a^2\partial q_b^2}. \tag{3.16}$$

We also know that if Z_0 is suitably renormalised (to $\bar{Z}$, by suitable choice of f) the Schwinger variational principle gives

$$\frac{\partial \bar{Z}}{\partial m_0^2}=-\frac{1}{2}\frac{\partial^2 \bar{Z}}{\partial q_a^2} \tag{3.17}$$

$$\frac{\partial \bar{Z}}{\partial \lambda_0}=-\frac{1}{4N}\frac{\partial^4 \bar{Z}}{\partial q_a^2\partial q_b^2} \tag{3.18}$$

of which, from equation (3.16), only one is independent.

In a more realistic theory these two equations (branching equations of the 'second kind' (Caianiello and Campolattaro 1962)) are related to the renormalisation group and Callan–Symanzik equations (Lowenstein 1974). The further demand that the N^{-1} expansion of the exact solutions satisfies these conditions individually order by order† is effected by imposing them upon the general solution for Z_0 of equation (3.1). Since $\bar{Z}_\pm$ of equation (3.2) individually satisfy equations (3.17) and (3.18) it follows that $\eta=\beta/\alpha$ is a function of N *alone*. It then follows that K of equation (3.10) cannot have a non-trivial asymptotic expansion in N^{-1} as λ varies. That is, boundary condition (ii) is excluded. Thus, although the N^{-1} expansion of § 2 can be derived from the Schwinger–Dyson equations alone, it is compatible with the branching equations of the 'second kind' (equations (3.17) and (3.18)) which we henceforth assume to be valid. Furthermore, from the small λ behaviour of $\theta(\lambda)$ it then follows that boundary condition (iii) cannot be valid for arbitrarily small λ.

To summarise, we have shown that the implicit boundary conditions of the N^{-1} expansion are not strong enough to determine the exact solutions for the connected Green functions. In consequence the N^{-1} expansions cannot *both* be uniquely resummable to an exact solution. Moreover, in the absence of further constraints (which will be imposed later), which solutions are, or are not, uniquely resummable depends on the unspecified boundary conditions. This failure of summability has nothing to do with the Borel summability of $(T_{2p}^\pm)_N$ but rather with the fact that the difference between the partial sums and the exact solutions (essentially proportional to K or K^{-1}) cannot be strongly enough bounded‡.

We have yet to impose 'physical' conditions upon these solutions. On doing so we find that the importance of Borel summability reappears. We postpone a discussion of these constraints and of Borel summability (discussed in part in Hikami and Brézin (1976)) until we have made a few comments about the λ-perturbation series.

4. The λ behaviour of Green functions and physical solutions

Having resolved the problem of the N^{-1} summability of the solutions to the Schwinger–Dyson equations we shall briefly discuss the λ behaviour of the exact solutions.

† If the approximations used for more realistic theories are not compatible with the renormalisation group there are considerable difficulties (Caianiello *et al* 1971). For realistic theories it seems that the N^{-1} expansion is compatible with renormalisation group equations.

‡ This was obscured in an earlier version of this work.

Assume that η is λ-independent (i.e. equations (3.17) and (3.18) are satisfied). Near $\lambda = 0$ (fixed N)

$$\tau_{2p}^{-}/\tau_{2p}^{+} = \mathrm{O}(\lambda^{-p-N/2}\, \mathrm{e}^{N/4\lambda}). \tag{4.1}$$

If we examine the possible asymptotic series in λ for τ_{2p} as η varies we find the expected two series.

(*a*)

$$\eta = 0 \qquad \tau_{2p} \sim (\tau_{2p})_\lambda = \frac{1}{(2m_0^2)^p}\frac{(\Sigma_{2p}^{+})_\lambda}{(\Sigma_0^{+})_\lambda} \tag{4.2}$$

with

$$(\Sigma_{2p}^{+})_\lambda = \sum_{n=0}^{\infty} \frac{\Gamma[(N+2p)/4+n]\Gamma[(N+2p+2)/4+n]}{\Gamma[(N+2p)/4]\Gamma[(N+2p+2)/4]}\left(-\frac{4\lambda}{N}\right)^n. \tag{4.3}$$

This is the usual λ-perturbation series†, Borel summable, and uniquely summable to an exact solution without having to consider any further constraints.

(*b*) $\eta \neq 0$, giving the non-λ-perturbative solution

$$\tau_{2p} \sim (\tau_{2p}^{-})_\lambda = \frac{(-1)^p}{(\lambda m_0^2)^p}\frac{\Gamma(N/2)}{\Gamma[(N+2p)/2]}\frac{(\Sigma_{2p}^{-})_\lambda}{(\Sigma_0^{-})_\lambda} \tag{4.4}$$

with

$$(\Sigma_{2p}^{-})_\lambda = {\sum_{n=0}^{\infty}}' \frac{\Gamma[(-N-2p)/4+n]\Gamma[(2-N-2p)/4+n]}{\Gamma[(-N-2p)/4]\Gamma[(2-N-2p)/4]}\left(\frac{4\lambda}{N}\right)^n \tag{4.5}$$

not uniquely summable to an exact solution. This lack of uniqueness is inevitable from the non-unique summability of $\bar{\tau}_{2p}^{+}/\bar{\tau}_{2p}^{-}$. We note that although $(\tau_{2p})_\lambda$ is not Borel summable when N is odd, for even N, it is a ratio of finite polynomials!

We do note that the situation with λ series is cleaner than with N^{-1} series, since there is no doubt as to which λ series is uniquely summable to an exact solution. In the absence of a further branching equation indicating variation with $\epsilon = N^{-1}$ we need an additional constraint to eliminate either boundary condition (iii) or (i) of the previous section and achieve the same end. As we have noted, to demand the same type of boundary condition valid for arbitrarily small λ eliminates (iii) and guarantees that the N^{-1} and λ expansions of the exact solutions are closely related. That is, if $(\tau_{2p})_N = (\tau_{2p}^{+})_N$ it follows that $(\tau_{2p})_\lambda = (\tau_{2p}^{+})_\lambda$‡ and if $(\tau_{2p})_N = (\tau_{2p}^{-})_N$ then $(\tau_{2p})_\lambda = (\tau_{2p}^{-})_\lambda$.

Having answered our initial questions on summability we wish to discuss briefly the possibility of finding physical reasons for the rejection of the non-unique solution ω_N^{-}. This might seem straightforward since ω_N^{-} corresponds to $\Gamma_2 = \chi^{-1}$ being negative (i.e. a

† We note that, assuming simple dominance of the heuristic path integrals by saddle-points, the perturbation series for massless O(N) invariant $(\phi^2)^2$ theory has (Brézin *et al* 1977, Itzykson *et al* 1977a,b, Parisi 1977)

$$(\Sigma_{2p}^{+})_\lambda = \sum_{n=0}^{\infty} a(n,p)\lambda^n$$

where

$$a(n,p) = n!(-1)^n n^\alpha \sigma^n D[1+\mathrm{O}(n^{-1})]$$

with σ independent of p, α dependent on p (independent of momentum) and D dependent on p.

‡ It is easily seen that $(\tau_{2p}^{\pm})_N$ and $(\tau_{2p}^{\pm})_\lambda$ have close partial sums. For example, for $N = p = 1$ the partial sums with m terms differ only by terms $\mathrm{O}(\lambda^{m+1})$.

'tachyon'). However, in order to eliminate this solution we must reject *all* of the exact solutions which it represents in the large N limit and, at least for some values of λ and η, these have positive Γ_2 in defiance of their asymptotic representative. In fact, if we consider Γ_2 in more detail we find the following. If $|\eta|>1$ Γ_2 is negative. If $|\eta|<1$ Γ_2 is positive for large λ and negative for small λ. If $\eta=0$ Γ_2 is everywhere positive. Curve A of each of figures 2–5 gives Γ_2 for varying values of η whilst curve B gives the leading order expansion in $1/N$ for the simple case $N=1$. Thus if we are to eliminate the non-unique solutions a more general form of this constraint must be found.

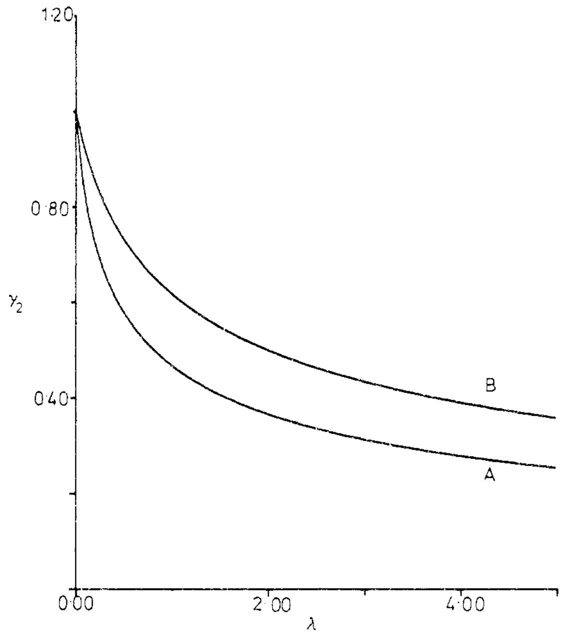


Figure 2. Curve A, the exact Green function $m_0^2\tau_2(\eta)$ for $\eta=0$. Curve B, the leading order approximation $m_0^2(\tau_2^+)_N$.

5. 'Physical' constraints

In this section we shall impose constraints appropriate to a regularised Euclidean field theory. The most relevant is that $Z[q]$ should satisfy positivity conditions (Symanzik 1964, Iliopoulos *et al* 1975).

A first consequence is that we must have

$$\tau_{2p}(\eta)>0 \qquad \forall p. \tag{5.1}$$

For $p=1$ this corresponds to eliminating tachyons. For $m_0^2>0$ it follows that, for (5.1) to be satisfied for large enough N, we must have $\eta=0$. That is, we have a unique solution. Thus, without any calculation we deduce that $(\tau_{2p}^+)_N$ is *uniquely* summable to a physical solution.

For $m_0^2<0$, condition (5.1) is not as restrictive, only requiring that $|\eta|\leqslant 1$. However, positivity also requires that

$$\sum_{i,j} c_i^* Z[\mathrm{i}(q_i-q_j)]c_j \geqslant 0 \qquad \forall c_i, \text{ real } q_i \tag{5.2}$$

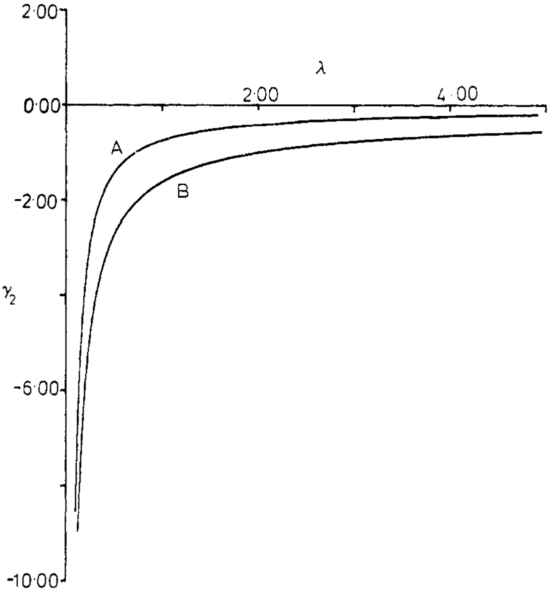


Figure 3. Curve A, the exact Green function $m_0^2\tau_2(\eta)$ for $\eta = 2$. Curve B, the leading order approximation $m_0^2(\tau_2^-)_N$.

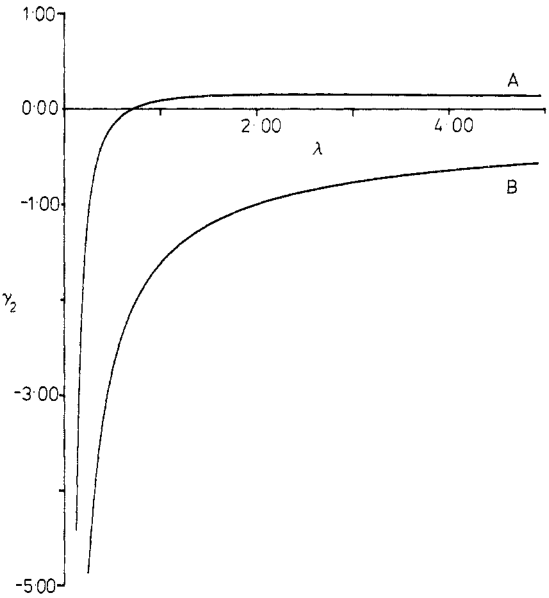


Figure 4. Curve A, the exact Green function $m_0^2\tau_2(\eta)$ for $\eta = \frac{1}{6}$. Curve B, the leading order approximation $m_0^2(\tau_2^-)_N$.

(where, without loss of generality in its application, we have taken $N = 1$ for simplicity). Taking $i, j = 1, 2$ gives

$$Z[\mathrm{i}q]+Z[-\mathrm{i}q]\leqslant 2Z[0]. \tag{5.3}$$

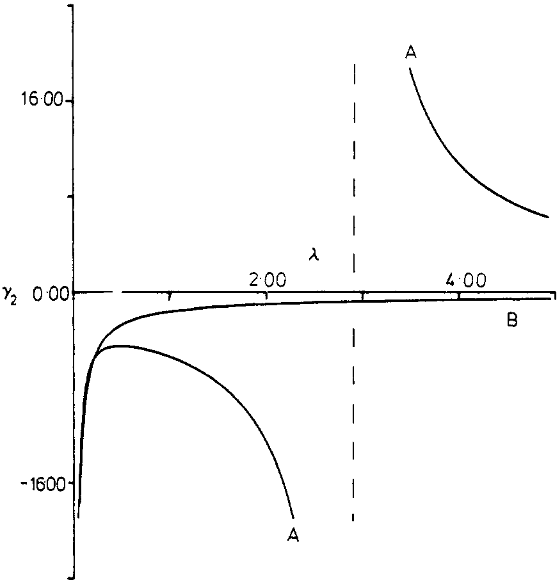


Figure 5. Curve A, the exact Green function $m_0^2\tau_2(\eta)$ for $\eta=-\frac{2}{3}$. Curve B, the leading order approximation $m_0^2(\tau_2^-)_N$.

This further restricts η to the range $-1\leqslant\eta\leqslant 0$. We are unable to restrict the value of η further by taking more than two c_i in (5.2).

Finally, we should impose convexity on $W[q]$. That is, we must have

$$\frac{\mathrm{d}^2W}{\mathrm{d}q^2}=\frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}q^2}-\left(\frac{1}{Z}\frac{\mathrm{d}Z}{\mathrm{d}q}\right)^2\geqslant 0. \tag{5.4}$$

This again gives $\Gamma_2\geqslant 0$ (and is always compatible with $\eta=0$) and by its non-linearity would be expected to be very restrictive. Unfortunately we are unable to prove analytically that $\eta=0$ is the only possibility. Assuming this, we shall examine the Borel summability of the N^{-1} expansions for Z (and compare them to the λ expansions) in the next section.

6. Borel summability

In determining the Borel summability of the N^{-1} expansion of

$$[\bar\tau_{2p}(m_0^2>0,\ \eta=0)]_N=(\bar\tau_{2p}^+)_N$$
$$[\bar\tau_{2p}(m_0^2<0,\ \eta=0)]_N=(\tau_{2p}^-)_N \tag{6.1}$$

it is helpful to consider the Borel summability of $Z^\pm(q_i=0)$ of equations (3.2) and (3.5).

Writing $Z^\pm$ from (3.5) as

$$Z^\pm(q_i=0)=\frac{2(\pi N)^{N/2}}{|m_0|^N\Gamma(N/2)}\int_{-\infty}^{\infty}\mathrm{d}t\,\exp(-NF_\pm(t)) \tag{6.2}$$

where

$$F_\pm(t)=\tfrac{1}{4}\lambda\,\mathrm{e}^{2t}\pm\tfrac{1}{2}\,\mathrm{e}^{t}-\tfrac{1}{2}t \tag{6.3}$$

(on substitution of $x^2 = N\,e^t$) it is straightforward to expand $Z^{\pm}$ about the leading saddle-points at $t_{\pm}(F'_{\pm}(t_{\pm}) = 0)$ where

$$t_{\pm} = \frac{\mp 1 + (1+4\lambda)^{1/2}}{2\lambda}. \tag{6.4}$$

The equation $F'_{\pm}(t) = 0$ has further (complex) solutions

$$t_{\pm}^{(n)} = \frac{\pm 1 + (1+4\lambda)^{1/2}}{2\lambda} + in\pi \qquad \text{integer } n. \tag{6.5}$$

The nearest of these ($n = \pm 1$) determine, via Darboux' theorem (Dingle 1973) the coefficients $A_K^{\pm}$ in the expansion

$$Z^{\pm}(q_i = 0) = \frac{2(\pi N)^{N/2}}{\Gamma(N/2)}\, e^{-NF_{\pm}(t_{\pm})} \left(\frac{2\pi}{NF''_{\pm}(t_{\pm})}\right)^{1/2} \left(1 + \sum_{1}^{\infty} \frac{A_K^{\pm}}{N^K}\right). \tag{6.6}$$

We have kept the notation of Hikami and Brézin (1976), in which A_K^{+} were calculated. In a similar manner it is straightforward to evaluate A_K^{-}. The reader is referred to Hikami and Brézin (1976) and Dingle (1973) for details and method of calculation. It is sufficient for our purposes to quote the combined result

$$A_K^{\pm} \sim A^{\pm} \frac{\Gamma(K+\frac{1}{2})}{K^{1/2}} \frac{\sin K\theta^{\pm}}{\rho^K} \tag{6.7}$$

where $A^{\pm}$ are irrelevant scale factors, and

$$\rho = (Z^2 + \pi^2/4)^{1/2} \tag{6.8}$$

$$Z = -\frac{(1+4\lambda)^{1/2}}{4\lambda} - \frac{1}{2}\ln\left(\frac{(1+4\lambda)^{1/2}+1}{(1+4\lambda)^{1/2}-1}\right) \tag{6.9}$$

$$\theta^{\pm} = \cos^{-1}(\pm Z/\rho). \tag{6.10}$$

That is, for all λ, the series $\Sigma A_K^{\pm} N^{-K}$ are Borel summable. The Borel transforms $B^{\pm}(b)$ have singularities at $|\arg b^{\pm}| = \theta^{\pm}$.

As we have noted, Z^- is *not* Borel summable (in general) as a power series in λ, the transform having singularities on the positive real axis. (Z^+ is Borel summable in λ, with singularities on the negative real axis).

This behaviour in λ is reflected in the behaviour of $\theta_{\pm}$ as $\lambda \to 0$. From (6.10) and (6.9) we see that, as $\lambda \to 0$,

$$\theta_{+} \approx \pi(1 - 2\sqrt{\lambda}) \qquad \lambda \approx 0. \tag{6.11}$$

(That is, the singularities of $B^+(b)$ pinch the negative b axis.)

On the other hand, for small λ,

$$\theta_{-} = \pi - \theta_{+} \approx 2\pi\sqrt{\lambda} \tag{6.12}$$

showing that, in this case, the singularities of $B^-(b)$ pinch the *positive* b axis, implying difficulty for the λ summability.

To summarise, we have seen that $(Z^+)_N$ is Borel summable (once the leading saddle-point exponential is factored out). This suggests that $(\tau_{2p}^{+})_N$ is the ratio of two Borel summable series, which is hardly surprising since τ_{2p}^{+} is a unique solution for $m_0^2 > 0$. For $m_0^2 < 0$ the situation is very different, since $(Z^-)_N$ is *still* Borel summable (after factoring out the saddle-point term) implying that $(\tau_{2p}^{-})_N$ is the ratio of Borel summable series. This is in contrast to the λ expansions for τ_{2p}^{-}, which (in general) are *not* Borel summable. Thus, rearranging the perturbation series as the N^{-1} expansion *does* convert a meaningless expansion into a more sensible one†.

The conclusions of this and preceding sections are summarised in tables 1 and 2.

7. The O(*N*) lattice model

We shall devote this final section to comments about the relevance of our conclusions for the single-mode (or zero-dimensional) model to a more realistic theory. As we have said before, we shall not be able to take infinite renormalisation into account. We therefore choose to work with an $O(N)$ theory on a finite lattice, with fixed lattice parameter a. Caianiello and co-authors (Caianiello *et al* 1978) have stressed the relevance of the single-mode approximation for lattice calculations and their ideas have been extended by other authors more recently (Bender *et al* 1979).

Let $Z[j_a]$ be the generating functional for the $O(N)$ invariant theory (2.1) on the lattice. The Schwinger–Dyson equation for Z is

$$(-\partial^2+m_0^2)\frac{\delta Z}{\delta j_a(x)}=j_a(x)Z-\frac{\lambda_0}{N}\frac{\delta^3 Z}{\delta j_a(x)\delta j_b^2(x)}. \tag{7.1}$$

If we write $Z[j_a]$ as

$$Z[j_a]=\left[\exp\left(-\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\frac{\delta}{\delta j_a(x)}K(x-y)\frac{\delta}{\delta j_a(y)}\right)\right]Z_1[j_a] \tag{7.2}$$

where

$$K(x-y)=-\partial^2\delta(x-y) \tag{7.3}$$

it follows from (7.1) that $Z_1[j_a]$ satisfies

$$m_0^2\frac{\delta Z_1}{\delta j_a(x)}=j_a(x)Z_1-\frac{\lambda_0}{N}\frac{\delta^3 Z_1}{\delta j_a(x)\delta j_b(x)^2}. \tag{7.4}$$

This is essentially the same equation as the equation (2.2) for $Z_0(q_a, m_0^2, \lambda_0, N)$ (or equivalently, equation (2.6) for $W_0(q_a, m_0^2, \lambda_0)$ of (2.5)). All our conclusions about the single-mode approximation can thus be transferred to the Schwinger–Dyson equation (7.4). Rather than reiterate them we want to go further and discuss the relationship of Z_1 to Z_0 and to Z, the generating functional for the full theory.

The exact relationship is most easily seen in terms of the generating functionals for connected Green functions. If

$$W_1(j_a, m_0^2, \lambda_0, N)=\ln Z_1(j_a, m_0^2, \lambda_0, N) \tag{7.5}$$

it follows that W_1 is expressible in terms of W_0 (equation (2.5)) by

$$W_1[j_a, m_0^2, \lambda_0, N]=\delta(0)\int \mathrm{d}x\, W_0(j_a(x)\delta(0)^{-1}, m_0^2\delta(0)^{-1}, \lambda_0\delta(0)^{-1}) \tag{7.6}$$

where $\delta(0)$ is shorthand for a^{-d} (for a d-dimensional lattice).

Thus, knowing W_0 enables us to determine W_1 directly. Inserting this in (7.2) we can develop a diagrammatic expansion (Bender *et al* 1979, Kövesi-Domokos 1976) for

† In these respects the $\hbar$ expansion resembles the λ expansion.

Table 1. The role of the iterative N^{-1} expansion scheme for $m_0^2 > 0$. The symbols 'US (NUS)' mean 'uniquely summable (not uniquely summable)'.

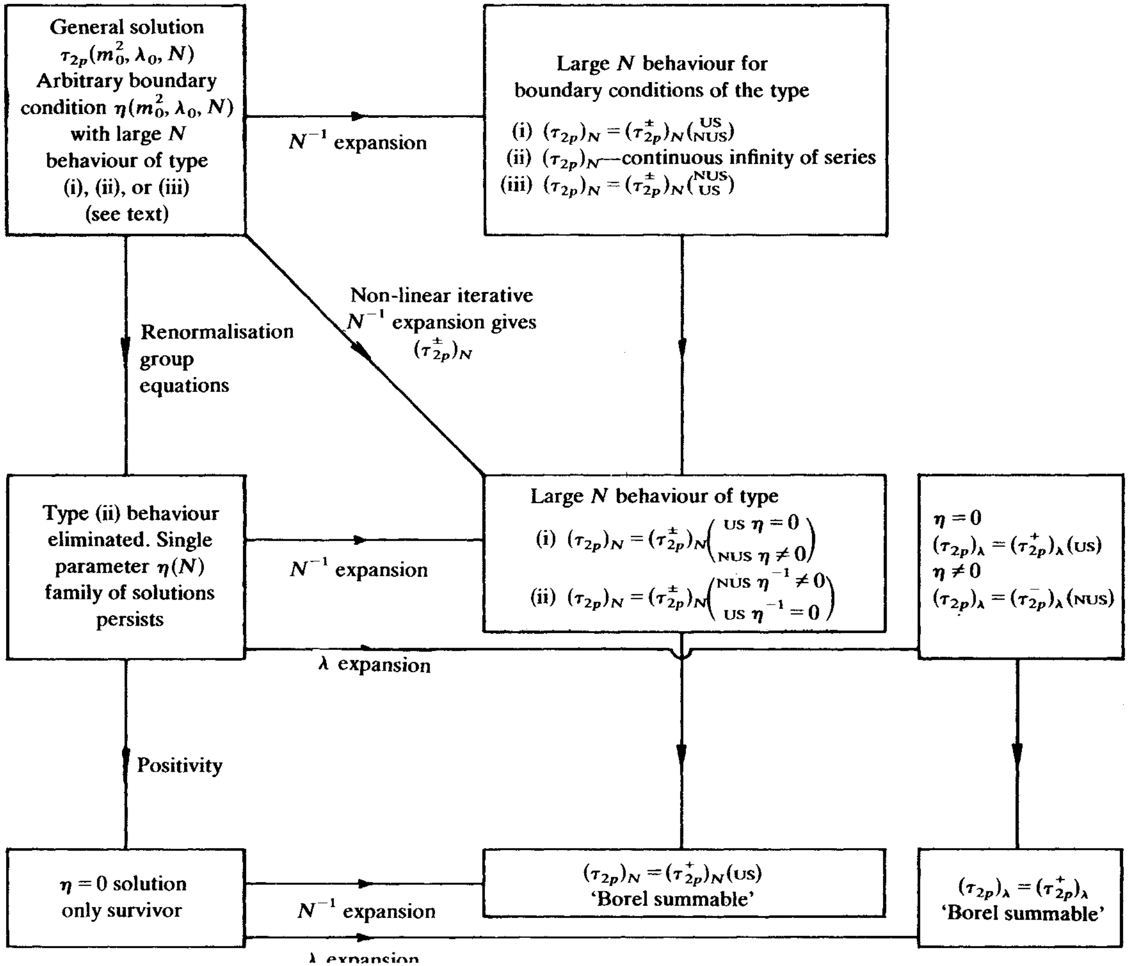
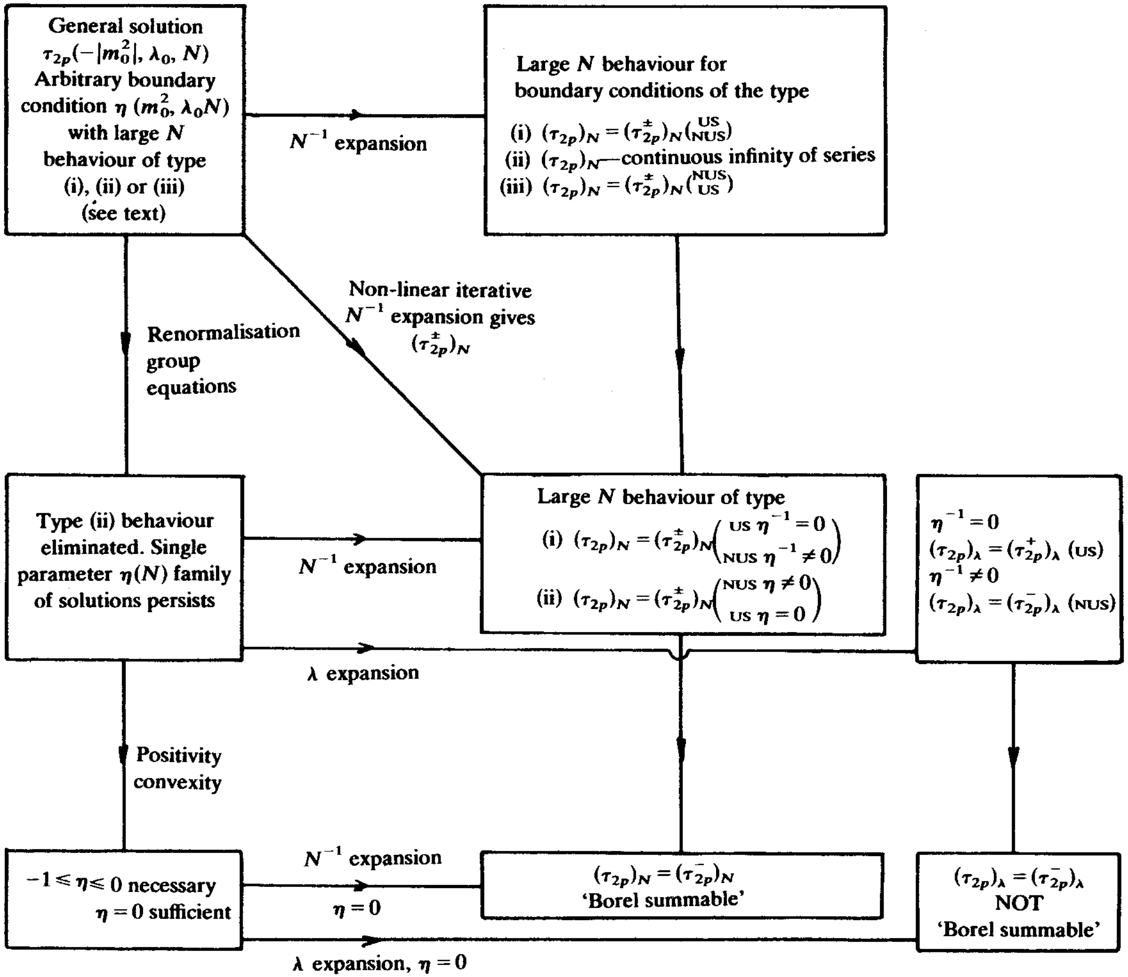


Table 2. The role of the iterative N^{-1} expansion scheme for $m_0^2 < 0$. $\tau_{2p}(-m_0^2, \eta) = (-1)^p \tau_{2p}(m_0^2, \eta^{-1})$.



$W = \ln Z$, with 'propagators' $\delta_{ab}K(x-y)$ and vertices

$$\lambda^{(n)}_{a_1 a_2 \ldots a_n} = \delta(0)^{1-n} \frac{\partial^n}{\partial q_{a_1} \ldots \partial q_{a_n}} W(q_a, m_0^2\delta(0)^{-1}, \lambda_0\delta(0)^{-1})\bigg|_{q_a=0}. \tag{7.7}$$

That is, $\lambda^{(n)}$ are directly expressible in terms of the Γ_n of equation (2.7) (and hence $\lambda^{(p)} = O(N^{1-p/2})$). It is thus straightforward to develop the N^{-1} expansion for W but we shall not pursue this here†.

Rather, we observe that, since

$$W_0(j_a\delta(0)^{-1}, m_0^2\delta(0)^{-1}, \lambda_0\delta(0)^{-1}) = W_0(j_a\delta(0)^{-3/4}\lambda_0^{-1/4}, m_0^2\delta(0)^{-1/2}\lambda_0^{-1/2}, 1) \tag{7.8}$$

we have

$$\lambda^{(n)} = O(\lambda_0^{-n/4}). \tag{7.9}$$

Since $\lambda^{(2p+1)} = 0$ the diagrammatic expansion is an expansion in $\lambda_0^{-1/2}$, and therefore appropriate for strong coupling.

This enables us to relate the 'physical' constraints of § 5 for the fake zero-dimensional theory to genuine physical constraints for the full (lattice) theory.

Suppose

$$\begin{aligned}\delta_{ij}\bar{\Gamma}_2 &= \frac{\partial^2 W_1(q_a, m_0^2, \lambda_0, N)}{\partial q_i \partial q_j}\bigg|_{q=0} = \delta(0)^{-1}\frac{\partial^2 W_0(q_0, m_0^2\delta(0)^{-1}, \lambda_0\delta(0)^{-1}, N)}{\partial q_i \partial q_j} \\ &= \delta_{ij}\delta(0)^{-1}\Gamma_2(m_0^2\delta(0)^{-1}, \lambda_0\delta(0)^{-1}, N)\end{aligned} \tag{7.10}$$

where Γ_2 is defined by (2.7).

The properties of Γ_2 were discussed in great detail in previous sections, and $\chi = \Gamma_2^{-1}$ was interpreted as the (mass)^2 of the scalar field of the zero-dimensional theory, whose positivity was necessary for this toy theory to be 'physical'. We shall see that $\bar{\Gamma}_2^{-1}$ has meaning in the full theory.

As an extreme example suppose, for the sake of argument, that we were to adopt the renormalisation prescription of Kövesi-Domokos (1976) in which all K loops vanish. The two-point function then has the form

$$G_{ij}(p^2) = \frac{\delta_{ij}}{p^2 - \bar{\Gamma}_2^{-1}}, \tag{7.11}$$

that is, $\bar{\chi} = \bar{\Gamma}_2^{-1}$ (related trivially to χ via (7.10)) is the *physical* scalar mass of the full theory if this renormalisation scheme were fully sensible. (Similar comparisons could be made between Γ_{2p}, $p = 1, 2, \ldots$, and physical quantities of the full theory.)

In fact, we believe that renormalisation is more complicated (Bender *et al* 1979) and this simple picture must be rejected. However, $\bar{\chi}$ still plays a role for large λ_0 (and fixed a) since the position of the pole in the two-point function is now at $p^2 = M^2$, where‡

$$M^2 = \bar{\Gamma}_2^{-1}(1 + O(\lambda_0^{-1/2})). \tag{7.12}$$

† This expansion in powers of the very singular K and powers of $\lambda^{-1/2}$ is not immediately reconcilable with conventional N^{-1} expansions which use K^{-1} and powers of λ (Schnitzer 1974a,b, Coleman *et al* 1975, Kobayashi and Kugo 1975, Abbott *et al* 1976, Rivers 1976). However, the leading behaviour is seen immediately to have common properties whichever expansion is chosen.

‡ To facilitate comparison between Bender *et al* (1979) and Kövesi-Domokos (1976) we note that $\bar{\Gamma}_2(m^2+\mu^2, \lambda)^{-1} = \bar{\Gamma}_2(m^2, \lambda)^{-1} + \mu^2 + O(\lambda^{-1/2})$.

Thus, for large λ_0, $\bar{\chi}$ (which is $O(\lambda_0^{1/2})$) is still the physical $(\text{mass})^2$ of the scalar field. Thus positive Γ_2 still implies the absence of tachyons (for large λ_0).

8. Conclusions

Our results for the single-mode (zero-dimension) approximation are displayed in tables 1 and 2. The significant feature is the relationship of the non-linear iterative N^{-1} expansions to the rest of the diagram. The left-hand column displays the general solution, the centre column the N^{-1} asymptotic series, and the right-hand column the λ-asymptotic series. We progress from top to bottom as more constraints are applied. Of particular note is the observation for $m_0^2<0$ that, even without determining necessary and sufficient conditions for $m_0^2<0$, the non-Borel summability of the λ expansion is not reflected in the N^{-1} expansion.

In the final section we have argued that the zero-dimensional model results can be extended in part to an $O(N)$-invariant scalar lattice theory. In particular, the 'physical' constraints imposed upon the zero-dimensional model are seen to be related to physical constraints on the full lattice theory.

In summary, we now understand the role of non-linear approximation schemes (e.g. Hartree–Fock-like approximations and their iterative extensions) in theories with no infinite renormalisation. We hope to be able to treat infinite renormalisation elsewhere, using the techniques of Bender *et al* (1979) for coming off the lattice.

Note added in proof. In a recent paper entitled *Path Integral Analysis of the Spontaneous Breakdown of Symmetry in a Static Ultralocal Field Theory* by S De Filippo and G Scarpetta (University of Salerno preprint) the authors considered some consequences of taking complex integration paths (in the x space of equation (3.2) for $N=1$) for $m_0^2<0$. While preserving the reality of W and forbidding 'tachyons' it is possible to have symmetry breaking. Whether this is sufficient is unclear to us. We thank the authors for communicating their results prior to publication.

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